Approximation of Decision problems

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Plan

1. **Approximations**
   - Randomized algorithms
   - Interactive Proofs, PCPs
   - Testers

2. **Distribution of inputs**
   - Average complexity,
   - Specific distributions
   - learning a distribution, machine learning

3. Streaming algorithms for community detection in social graphs: **combining 1 + 2** (Joint work with Claire Mathieu)

   Open problems

1. Approximations: a source of algorithms

- Randomized algorithms 1970s

If x $\epsilon L, then\ Prob\ [A(x)\ \ accepts] \geq \frac{2}{3}$

If $\neg$ (x $\epsilon L), then\ Prob\ [A(x)\ rejects] \geq \frac{2}{3}$

- Interactive Proofs, PCPs 1980-90s

- Testers 2000s

If x $\epsilon L,\ \ then\ Prob\ [A(x)\ \ accepts] = 1$

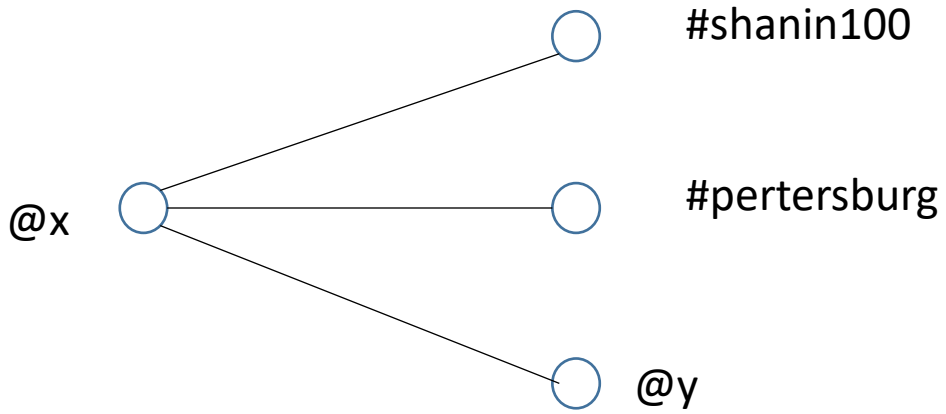
If x $\varepsilon-$far from $L, then\ Prob\ [A(x)\ rejects] \geq \frac{2}{3}$

Bigdata (since 2010)

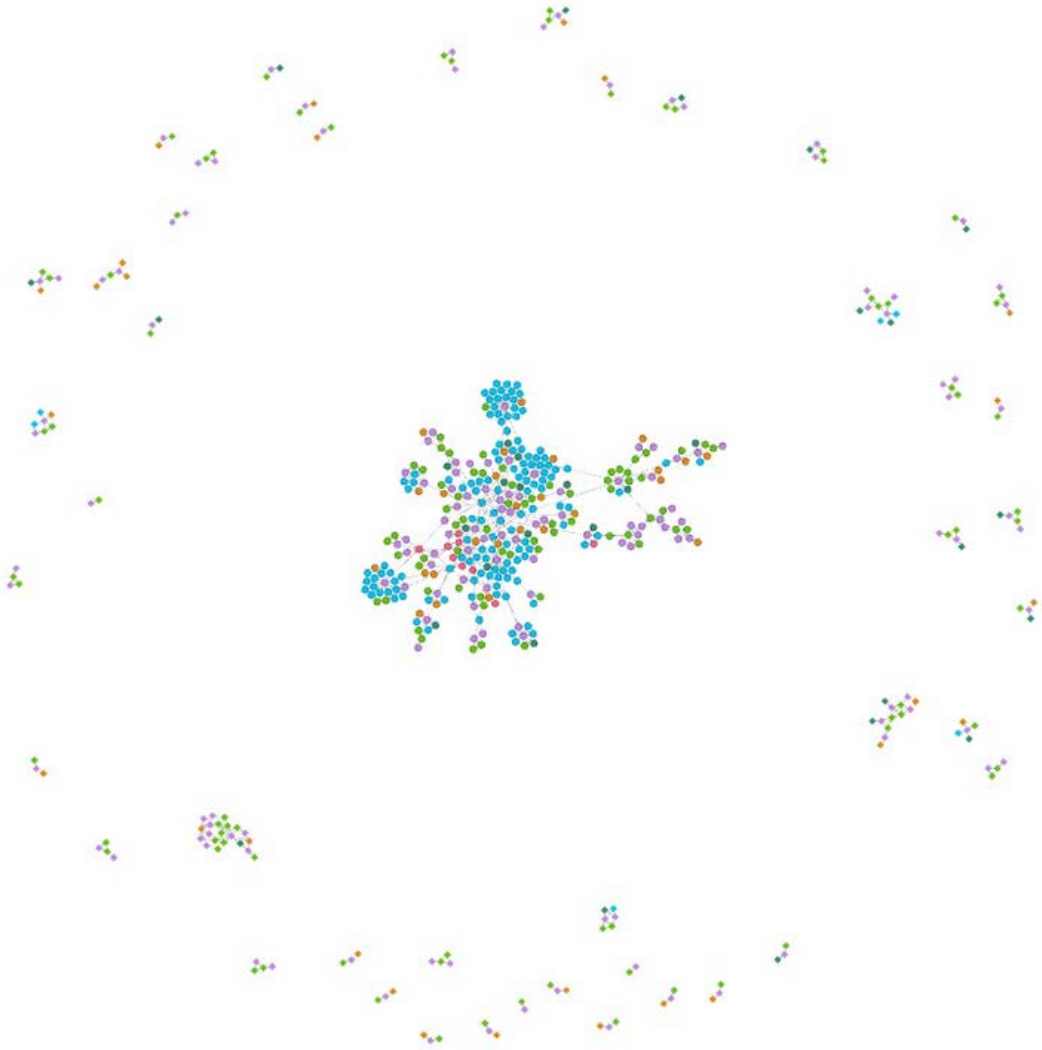
Volume, Velocity, Veracity, Variety

Social Networks: streams of edges

Twitter: @x send I'll give a talk at #shanin100 in #petersburg with @y



Keyword: #bitcoin, stream of tweets, of edges (10^3/min)



Is a graph G 3-colorable ?

3COL: assign some color to each node, no unicolor edge

- NP-complete, Testable (GGR 98) with $O(1/\varepsilon^2)$ queries
  - ε -noise : $\varepsilon.n$ insertions or deletions (Edit distance)
  - Sample on dense graphs: k random vertices, G_k the random subgraph

Let $G' = G + \varepsilon$

How can we decide if G is 3COL ?

- If G' is $2.\varepsilon$-far from 3COL, then G is at least ε-far from 3COL.

Does the graph G has a large γ-clique ?

Problem γ-LS: existence of a large dense subgraph , i.e. a set large subgraph (S,E(S) such that $|E(S)| > \gamma . |S|^2$

- NP-complete,
- Most Cluster detection algorithms use the Conductance

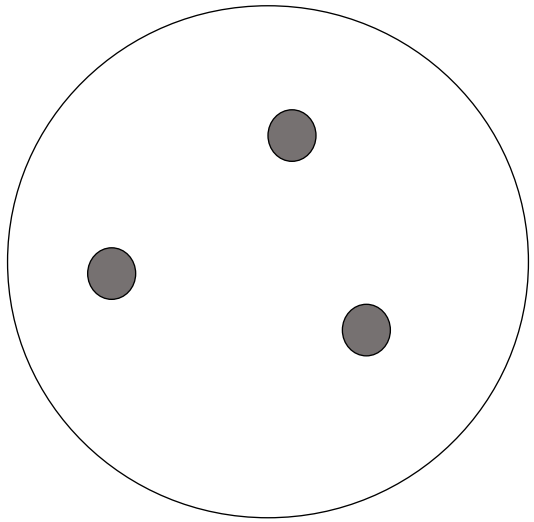
Let G'=G+ ε

How can we decide if G has a large γ-clique ?

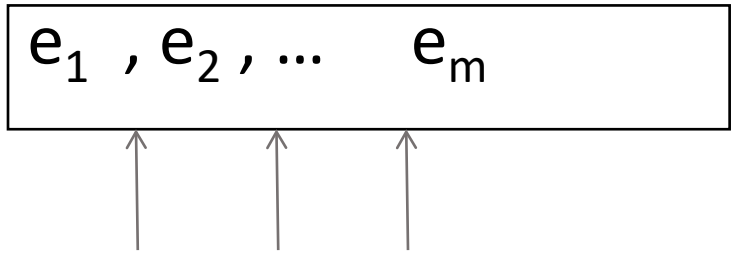
- Samples : uniform edges in a stream
- Arguments: Giant components in random graphs

Graphs and Streaming edges

- Samples in dynamic sparse graphs
- Global graph versus a stream of edges:



Graph and Samples



Stream and Samples

Samples

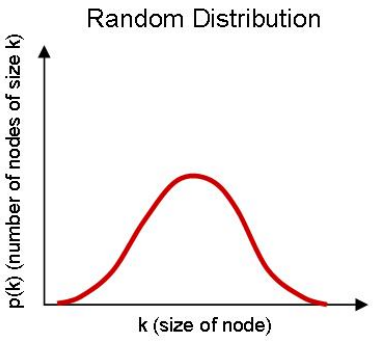
- **Fixed graphs**: let r random nodes *proportional to their degree*
- Katzir's method: $\varphi_1 = \sum_{i=1}^{r} \mathrm{d}_i$ $\varphi_{-1} = \sum_{i=1}^{r} \frac{1}{\mathrm{d}_i}$ $C =$ number of collisions

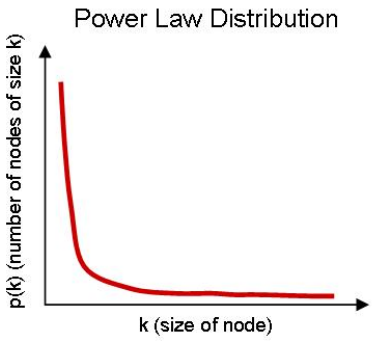
  R= $\varphi_1 . \varphi_{-1}$ -r

  $\frac{R}{C}$ approximates n, the size of the graph
- **Streaming:** count m the number of edges, $\frac{m}{d_{average}}$ approximates n
- Uniform edges yield random nodes proportional to their degree
- Reservoir sampling R(k): each edge has probability k/m to be in R.

2. Distributions on graphs

- Erdos-Renyi: G(n,p)
- Preferential Attachment: PA(n,m)
- **Configuration Model (Any Degree distribution)**





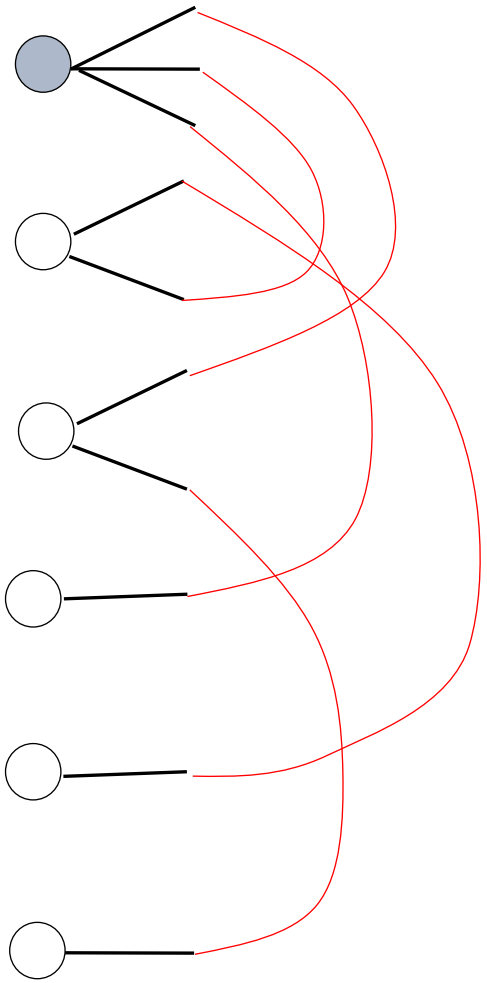
m=n.log n

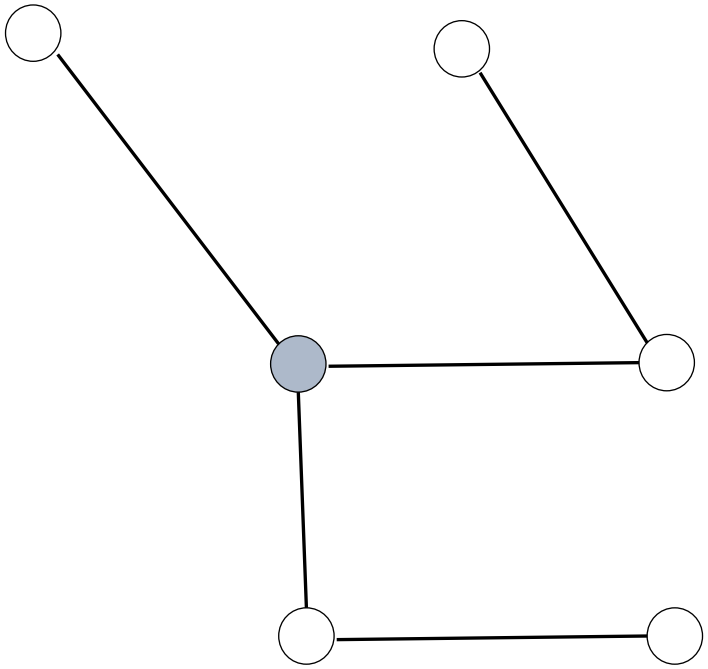
$d_{max} = \sqrt{c.n}$

Social graphs follow a Power law: $\text{Prob}_{\Omega}[\text{ degree}(x)=i\ ] = c/i^2$

Bigdata follow specific distributions

Configuration model: (3, 2, 1): random $\pi(i)=j$





3. Algorithm to detect γ-cliques

- **Hypothesis:** graphs follow a power law degree distribution, large S

- Streaming probabilistic **Detection Algorithm(k,m, ϒ):**
  1. Sample edges with a Reservoir (k) (probability $\frac{\mathrm{k}}{\mathrm{m}} > \sqrt{a/n}$)
  2. If largest connected component C has size > c_0 then Yes else No

Theorem 1: If m is large enough and $|S| > \frac{m}{\Upsilon.k}$, then:

(a) If S is a ϒ-clique then $\mathrm{Prob}_{\boldsymbol{\Omega}}[\text{Detection Algorithm Accepts}] > 1 - \delta$,

(b) If there is no large ϒ-clique $\mathrm{Prob}_{\boldsymbol{\Omega}}[\text{Detection Algorithm Rejects}] > 1 - \delta$

Giant component and phase transition

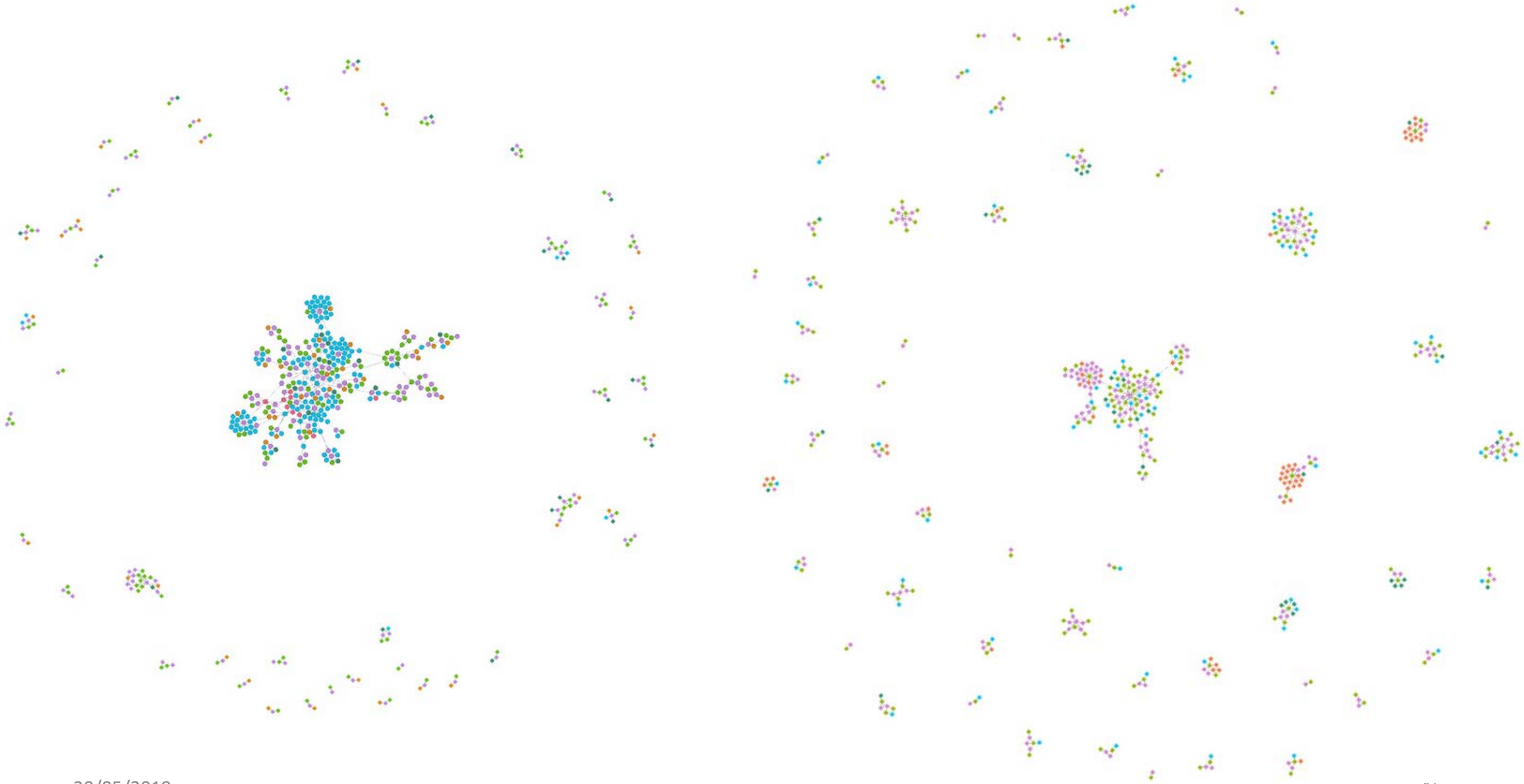
- Erdos-Renyi: G(n,p) $p>\frac{1}{n}$ giant component
- Configuration Model $\mathbf{E}[\boldsymbol{D}^{\mathbf{2}}]-\mathbf{2.E}[\boldsymbol{D}\quad]>0$ giant component (Molloy-Reed 2010)
- γ -clique: G(n,p) **Result 1:** $p>\frac{1}{\gamma.n}$ then giant component
- Proof of theorem 1 (a) |S|=n $\frac{k}{m}>\frac{1}{\Upsilon.n}$ and |S|=n> $\frac{m}{\Upsilon.k}$

Proof of theorem 1 (b)

- Use the statistical hypothesis
- Study the uniform configuration model
- Use fundamental results on random graphs (Molloy-Reed)

- Conclusion: almost surely, there is no giant component.

Giant components k=500

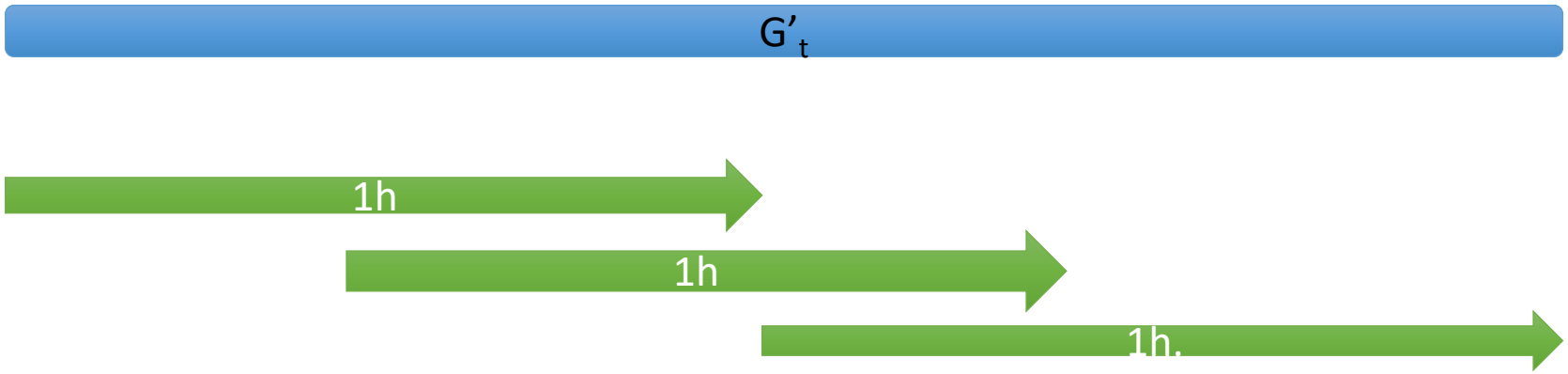


Generalization to dynamic graphs

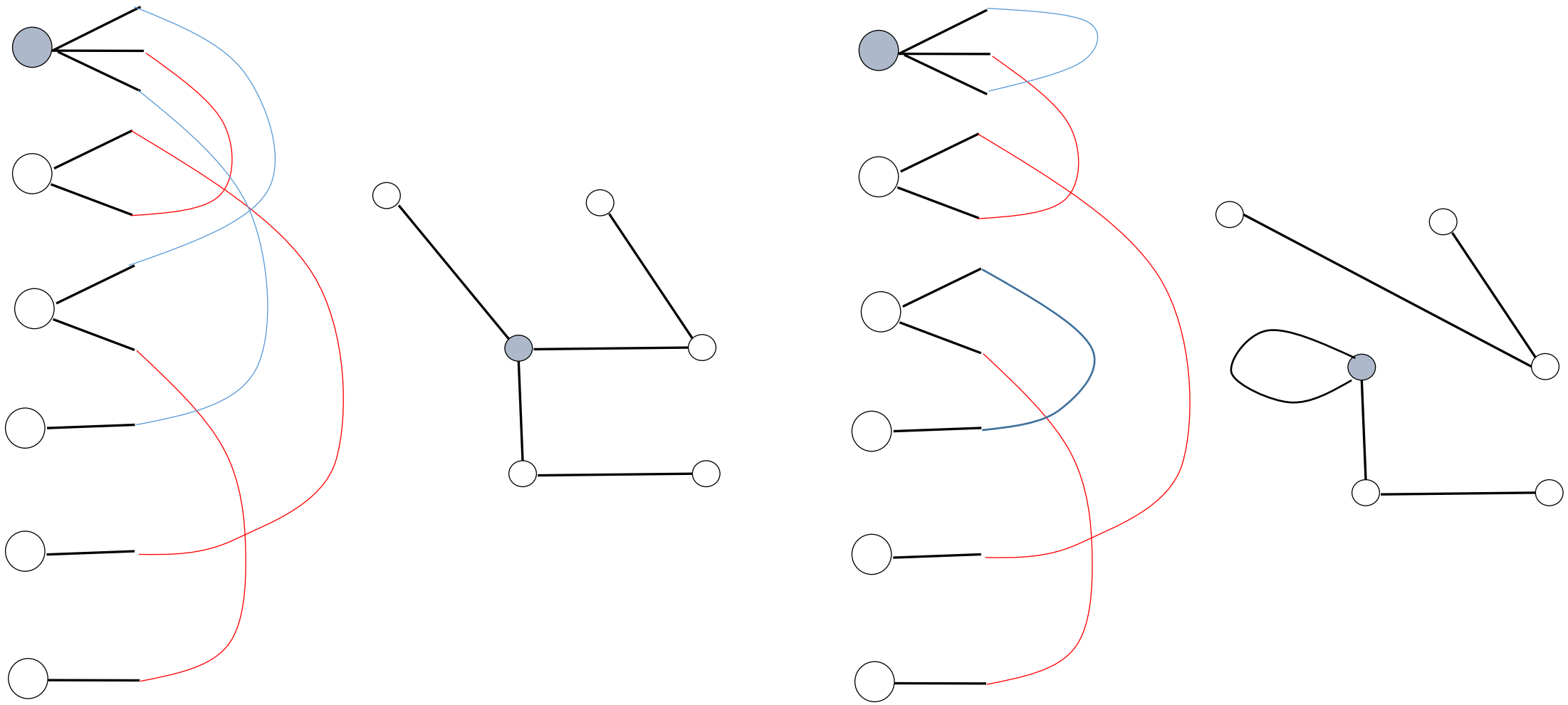
- Sliding windows
- Erdos-Renyi G(n,p): non-edges appear (q), edges disappear (r)
- Preferential Attachment: PA(n,m), no edge removal
- **Configuration Model:** some edges disappear, some appear, the statistical degree distribution is fixed.

Sliding Windows

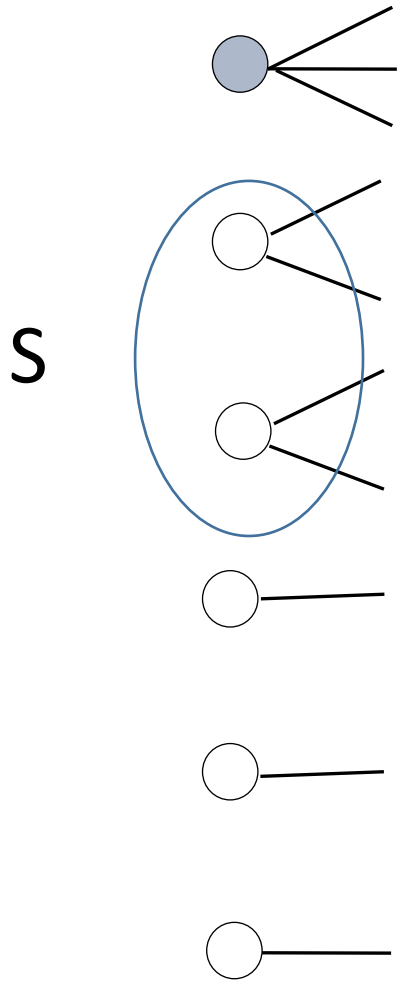
Assume a stream of edges with timestamps: e_1 , e_2 , … e_n ….



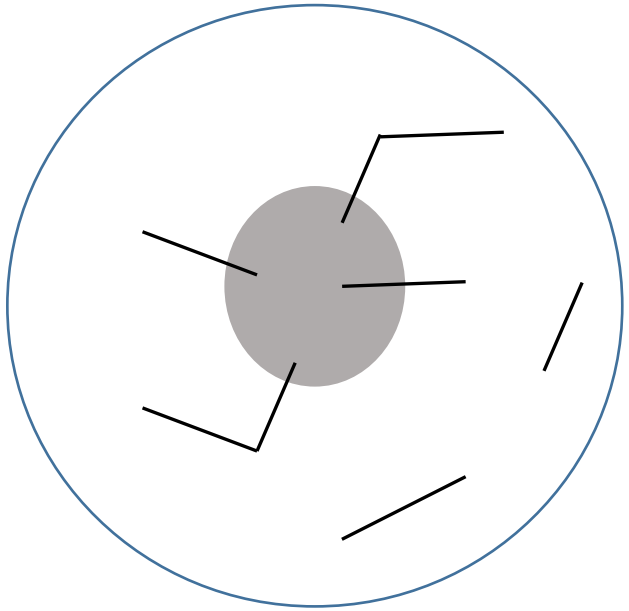
Uniform Dynamics : remove random q edges, rematch



S-concentrated model



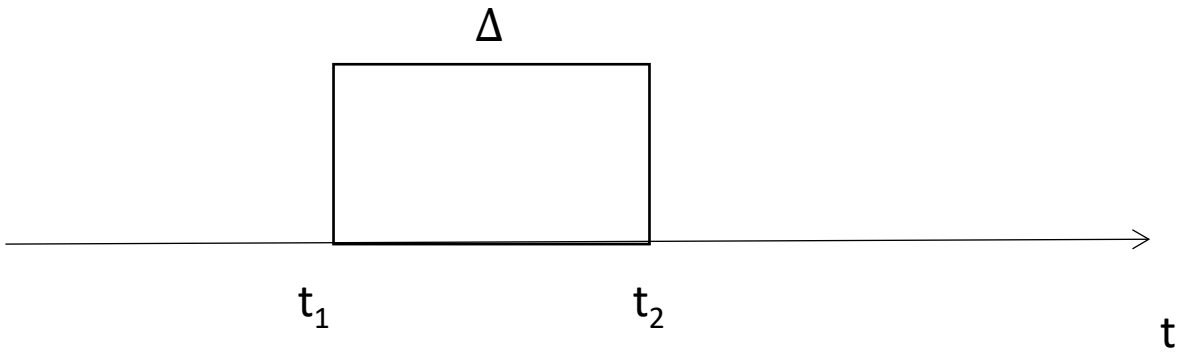
$\pi(i)=j$ for i,j in S ($p=\Upsilon$),
$\pi'(i)=j$ for i,j in V-S



Υ -clique

General Dynamics

Decide at any given time, if you choose the uniform dynamics or the S-concentrated dynamics.



Step dynamics:
Uniform dynamics until t_1
S-concentrated dynamics until t_2
Uniform dynamics after t_2

Step dynamics of length Δ for windows of length τ

Detection algorithm for the dynamic version

Temporal properties for general dynamics

◊ P : there is a t such that G_t satisfies P

□ P : for all t G_t satisfies P

Dynamic detection Algorithm. Let C(t) the largest connected component of R(k) at time before t. If $|C(t)| > c_0$, then Accept else Reject.

Theorem 2: If m is large enough and $|S| > \frac{m}{\Upsilon.k}$,

(a) For the S-step , $\mathrm{Prob}_{\boldsymbol{\Omega}}[\text{Dynamic Algorithm Accepts}] > 1 - \delta^{\Delta/\tau}$,

(b) For the uniform dynamics $\mathrm{Prob}_{\boldsymbol{\Omega}}[\text{Dynamic Algorithm } Rejects] > 1 - t.\delta/\tau$

Correlation of two streaming graphs

Several streams: S_1, S_2,……

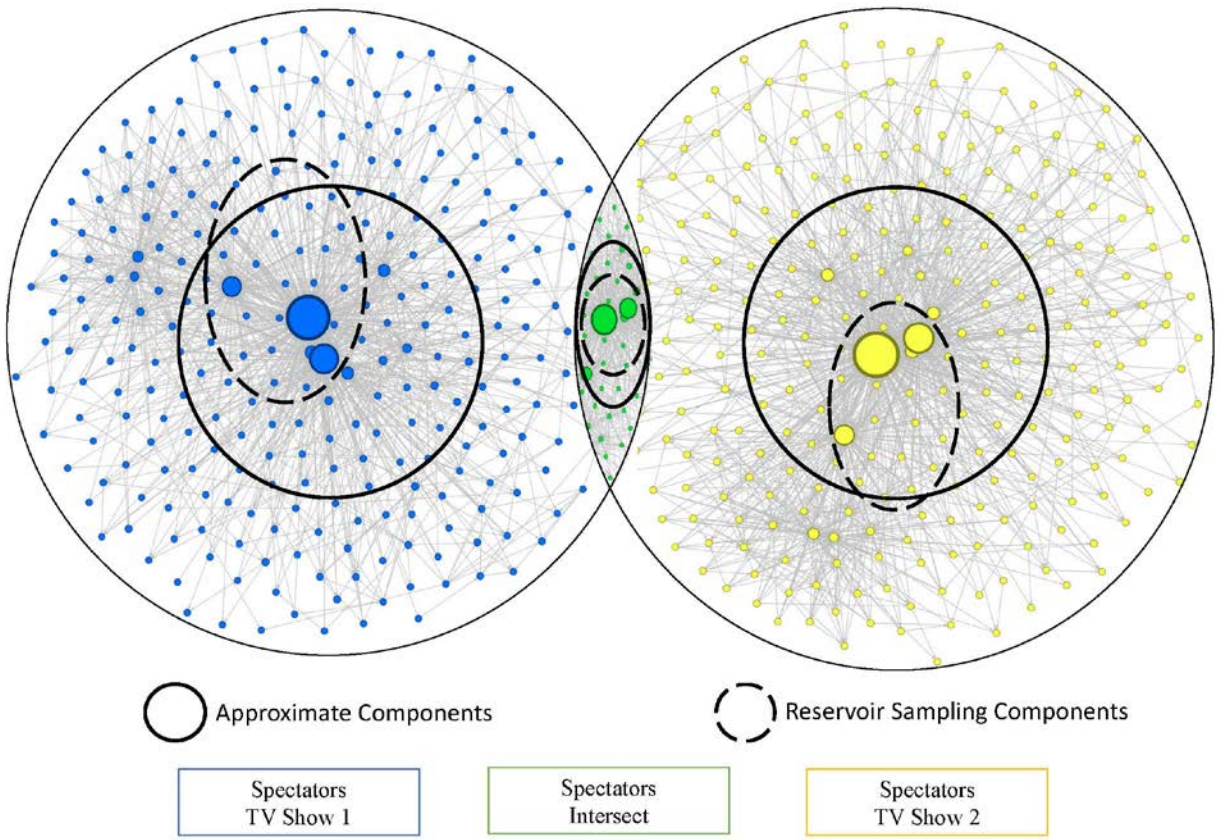
S_j : we compress the stream to C_j the union of the large connected components $C_j = \bigcup_i C_{i,j}$

The correlation between two streams is:

$$\rho(S_1, S_2) = J(C_1, C_2)$$

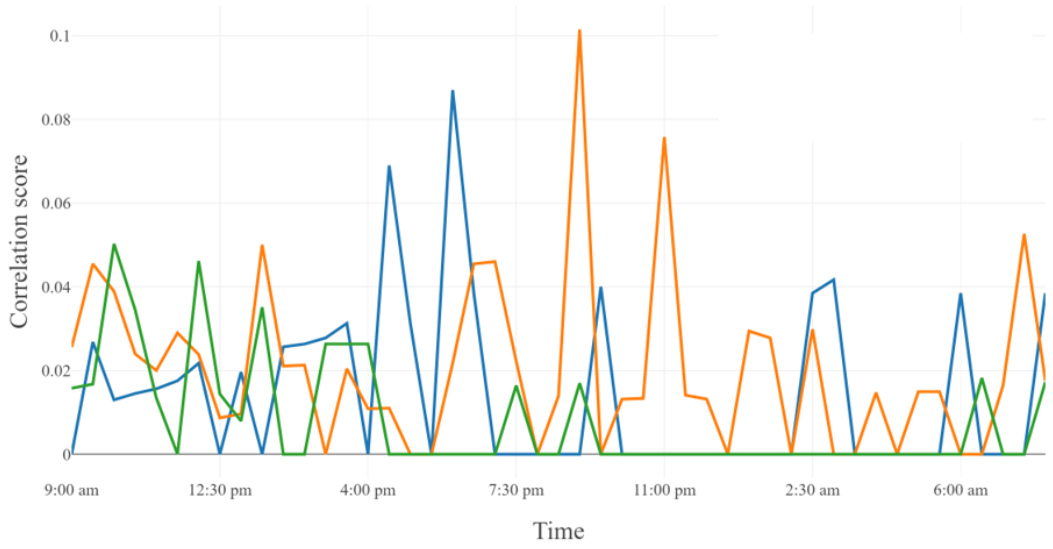
$J(C_1, C_2)$ is the Jaccard similarity (size of the intersection/ size of the union)

Two streams

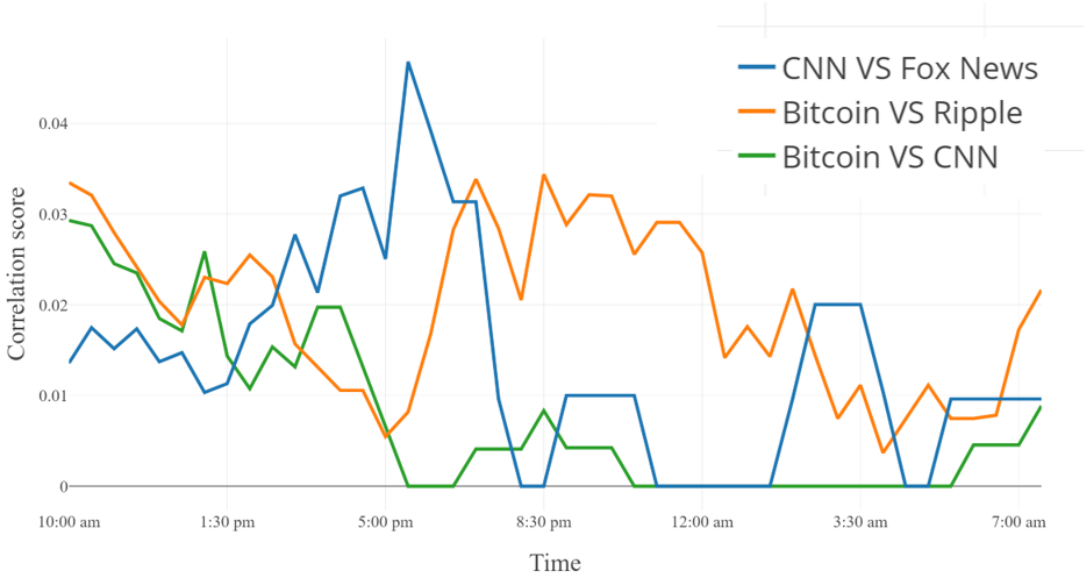

Approximate Components
Reservoir Sampling Components
Spectators
TV Show 1
Spectators
Intersect
Spectators
TV Show 2

Correlations between 3 pairs

Correlation



Averaged correlation



Search by correlation

1. Correlation between Streams of social graphs, based on their giant components

2. Distance between tags

3. Search by correlation: given tags σ_1 , σ_2 ….. σ_k find the closest tags.

Open problems: Explainability

1. **Search by correlation:** how to explain that a tag is not in the output list of the search?
2. Algorithm with a social function: admission to the University, to a service,…

   Explaination for a student not admitted ?

3. Privacy. GDPR is a European standard for platforms. How to verify that a platform follows or is far from satisfying the GDPR?

Conclusion

1. Social graphs are dynamic data with error

2. They follow some statistical model: learning algorithms offer a representation for some distributions

3. Is the noise a problem or an asset?

4. Are good algorithms simple?

5. Explainations of an algorithm